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How to Build and Coach a College Math Team

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Abstract. In this article, I am describing my experience as a math team coordinator at University of Rochester, which includes, among others, organizing undergraduate problem solving classes and seminars, conducting outreach activities for middle school and high school students, and actively recruiting college prospects.

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1 Introduction

After my last participation as a high school student in a mathematical competition, the 1992 Annual Mathematical Gazette Contest, I promised myself that I would take the first opportunity to train students for such events. Prior to coming to Rochester, I advised, on an individual basis, a series of talented competitors, both in Romania and in the United States. However, at my current institution, I have worked in a different role, acting as the faculty advisor for the math team program.

First of all, what is a college math team program? As its name suggests, it is an extracurricular activity aimed at students with an interest in mathematics and its applications, who would like to pursue further certain ideas encountered in traditional mathematics classes. Usually, this includes problem solving sessions or seminars, run by a faculty member or a group of students, and participation in mathematical competitions.

Each academic year, the main event of a college math team is the participation in the William Lowell Putnam Mathematical Competition [1], [2], which is the premier undergraduate math contest in the United States and Canada. This contest started in 1938, being organized by the Mathematical Association of America. In 2008, a total of 3627 students from 545 colleges took the exam, with 405 officially registered teams. The competition is comprised of two 3-hour sessions in which the participants have to attempt individually a set of 6 problems per session, each problem being worth 10 points. The majority of the problems requires nothing more than what is considered to be basic college mathematics: calculus, multivariable calculus, and linear algebra with differential equations. However, it is the extremely creative way in which one needs to put together this knowledge to solve the problems, that makes this test very difficult. In support of the previous statement comes also the fact that the median score is usually 1 or 2 (out of 120) [1], [2].

Any interested student at a participating college can take the exam. Although, to register a team, a university must designate in advance three students whose individual rankings will count towards the team placement. The lower the sum of the three individual rankings, the higher the team is in the final standings. It happened quite a few times that a certain school had remarkable individual performances, but a not so high team ranking. This happened because the students on the official team were outperformed by the ones not on the team. On one occasion, Harvard University had four of the best five individual scores, yet its team was only ranked fourth. This proves that the selection of the team is not a trivial matter, a faculty advisor having to take into consideration a lot of factors.

The Putnam exam attracts quite a few students, because, on top of the instant fame and the consistent monetary prizes awarded both in the individual and the team competitions, a number of graduate programs uses the top 500 list for recruiting purposes.

2 The Problem Solving Seminar

The first step in building a competitive math team is to identify students with above average mathematical abilities. The curriculum at University of Rochester offers, in conjunction with traditional versions of mathematics courses, honors classes aimed at mathematically talented students, that emphasize both theoretical understanding and mastering of technical skills. Students that took these type of courses were the first ones invited to join the math team program. However, as the word spread out and the interest grew, undergraduates with various backgrounds (e.g., physics, English, and music majors) attended the program.

In my first semester at Rochester, I started organizing a weekly problem solving seminar emulating the Romanian math circles I was a part of in my school years. A typical lecture on a certain topic would include in the beginning a presentation of the theoretical facts, followed by an open discussion of a related set of problems. The idea was to choose topics that were at the same time interesting, slightly touched in regular classes, and relevant to the Putnam competition. Such a theme is recurrence relations, that could have been presented in a formal way during a course in discrete mathematics or real analysis. Just in the last five years, the following problems appeared on the Putnam exam:

Problem A3 (2004). Define a sequence $\{u_n\}_{n=0}^{\infty}$ *by* $u_0 = u_1 = u_2 = 1$*, and thereafter by the condition that*

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$*. Show that* u_n *is an integer for all* n*. (By convention,* $0! = 1$*.)*

Problem A3 (2006). Let $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$ *be a sequence defined by* $x_k = k$ *for* $k = 1, 2, \dots, 2006$ *and* $x_{k+1} = x_k + x_{k-2005}$ *for* $k \geq 2006$*. Show that the sequence has 2005 consecutive terms each divisible by 2006.*

Problem B6 (2006). Let k *be an integer greater than 1. Suppose* $a_0 > 0$*, and define*

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Problem B4 (2007). Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a\rfloor$ means the largest integer $\leq a$.)

Problem B2 (2008). Let $F_0(x) = \ln x$ and $F_{n+1}(x) = \int_0^x F_n(t)\,dt$, for $n \geq 0$ and $x > 0$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

Other themes discussed during that academic year included: the Stolz - Cesaro, the intermediate and the mean value theorems, applications of complex numbers to polynomials, Diophantine equations, and geometric probability.

Following the success of this initiative and to attract more students, the department allowed for the seminar to be taught as a 1-credit fall semester class, "Topics in Problem Solving", that could be taken up to two times by an undergraduate. This is usually the case also at other competitive math programs: MIT, University of California at Berkeley, and Harvey Mudd College. The structure of the course is identical to the one of the seminar. The only modification was that, for grading purposes, students had to turn in throughout the semester the solutions to 15-20 problems from the homework sets that were assigned. The topics discussed in this class featured other interesting subjects that lend themselves to problem solving: the pigeonhole principle, functional equations, classical inequalities, the invariance principle, the floor function, congruences and divisibility, generating functions.

3 Mathematical Competitions

A math team advisor is faced each fall with the question of who are the students that should be included on the official Putnam team roster. The registration for both individual entries and the official team is relatively early, in the middle of October, with an eventual modification of the team to be submitted by late November. In part, due to these deadlines, a good number of schools uses previous Putnam scores to decide the composition of the team. However, this can lead to situations as the one described for Harvard, or, more recently, MIT. Personally, I think a past result does not tell the full story about the current form of a mathlete. Also, anyone can have a less-than-stellar performance on a certain day.

This is why I decided to administer a couple of selection tests: one that is given before the October deadline, and another one a month later that would clarify whether any modifications are needed. I believe this to be a more optimal process, as the students would have a longer training period through the problem solving meetings. The first selection test, with a time limit of $1\frac{1}{2}$ hours, contains 3 problems and is usually administered in September. In 2006, the students had to solve the following problems:

1. Let us consider 10 segments whose lengths are in between 1 and 55. Prove that we can choose 3 of them to be the sides of a non-degenerate triangle.

2. Let p be a prime number, $p \equiv 5(mod\ 6)$, and a, b two integers such that p is a divisor of $a^2 + ab + b^2$. Prove that a and b are both divisible by p.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying

$$f(f(x)) = af(x) + bx, \quad (\forall)x \in \mathbb{R}$$

where a, $b \in (0, 1)$. Prove that $f(0) = 0$.

For the second selection test I used the Virginia Tech Regional Mathematics Contest [3], organized each fall since 1979 by the Mathematics Department at Virginia Tech. The participants take, in late October, a 7 problem $2\frac{1}{2}$-hour test at their own colleges, under the supervision of a faculty member. 300 students from 50 colleges (the majority from the East Coast) compete in a typical year. Among the participating schools there are names with a strong record in mathematical competitions: Princeton University, Duke University, University of Michigan, and Carnegie Mellon University. Though built at a faster pace than the Putnam, the Virginia Tech exam is friendlier in terms of the level of the problems, some of them requiring only a solid high school mathematical background. This is one of the reasons I strongly encourage the freshmen and the sophomores to consider it, in preparation for the Putnam. Here are the problems of the 2008 contest:

1. Find the maximum value of

$$xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$$

where $0 \le x, y, z \le 1$.

2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

3. Find the area of the region of points (x, y) in the xy-plane such that

$$x^4 + y^4 \le x^2 - x^2y^2 + y^2$$

4. Let ABC be a triangle, let M be the midpoint of BC, and let X be a point on AM. Let BX meet AC at N, and let CX meet AB at P. If $\angle MAC = \angle BCP$, prove that $\angle BNC = \angle CPA$.

5. Let $(a_n)_n$ be a sequence of nonnegative real numbers and let π, ρ be permutations of the positive integers $\mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} a_n = 1$ and ϵ is a real number such that

$$\sum_{n=1}^{\infty} |a_n - a_{\pi(n)}| + \sum_{n=1}^{\infty} |a_n - a_{\rho(n)}| < \epsilon$$

Prove that there exists a finite subset X *of* $\mathbb{N}$ *such that*

$$\min\{|X \cap \pi(X)|, |X \cap \rho(X)|\} > (1-\epsilon)|X|$$

Here $|X|$ *indicates the number of elements in* X*.*

6. Find all pairs of positive (nonzero) integers (a, b) *such that* $ab-1$ *divides* $a^4 - 3a^2 + 1$*.*

7. Let $f_1(x) = x$ *and* $f_{n+1}(x) = x^{f_n(x)}$ *for* n *a positive integer. Thus* $f_2(x) = x^x$ *and* $f_3(x) = x^{x^x}$*. Now define*

$$g(x) = \lim_{n\to\infty} \frac{1}{f_n(x)}, \qquad (\forall)x > 1$$

Is g *continuous on the open interval* $(1, \infty)$*? Justify your answer.*

Usually, the Putnam and the Virginia Tech competitions take place in a span of less than two months, during the fall semester. However, I wanted to maintain a decent level of activity for my mathletes throughout the spring semester, in order to keep them in shape. The idea that I came up with, apart from regular seminar meetings, was to organize a regional spring contest, along the lines of the Virginia Tech one, entitled "University of Rochester Mathematical Olympiad".

We started first by inviting only other local area colleges, including Cornell University (another school with strong performances in the Putnam contest: nine top 5 finishes), to gauge the interest in such a competition. The feedback that we received was really promising, and so, for the last edition (the third one), we ended up with a roster of 96 students from 13 New York State colleges. I designed this olympiad to have a much more relaxed format: only 4 problems to be solved during a 3-hour period. This is such that the time does not play a big role and the students have a real chance to try and think about all the problems on the test. The problems of the March 2009 olympiad were as follows:

1. Let n *and* k *be positive integers. An* n*-digit whole number*

$$X = \overline{A_1 A_2 \ldots A_n}$$

is called k*-transposable if*

$$k \cdot X = \overline{A_2 \ldots A_n A_1}$$

Prove that there exists only two 6-digit 3-transposable (i.e., $n = 6$ *and* $k = 3$*) numbers and find them.*

2. Let $\triangle ABC$ *be an equilateral triangle and consider* M *a point on the segment* BC*, which is not its midpoint. The line perpendicular in* M *on* BC *intersects the parallel drawn through* A *to* BC *in the point* O*. The circle centered at* O*, having radius* OM*, intersects the sides* AB *and* AC *in the points* P*, respectively* Q*. Prove that* A*,* O*,* P*, and* Q *lie on the same circle.*

3. Prove that for n distinct positive integers $x_1, \ldots, x_n$, the following inequality holds:

$$(x_1 + x_2 + \ldots + x_n)^2 \le x_1^3 + x_2^3 + \ldots + x_n^3$$

4. $\lfloor x \rfloor$, also called the floor function, is defined to be the largest integer not greater than x. For example, $\lfloor \pi \rfloor = 3$, while $\lfloor -1.4 \rfloor = -2$. Find, with proof,

$$\left\lfloor \sum_{k=1}^{2^{2009}} k^{\frac{1}{2009}-1} \right\rfloor$$

Having a calendar of mathematical competitions like this one helps a lot in building each year both the level of knowledge and experience for the members of our math team program. This is reflected by the competitive rankings of our students obtained in the past three academic years:

	2006-2007	2007-2008	2008-2009
Putnam exam (team)	61	32	10
Putnam exam (individual)	266, 390	113	145, 157, 262, 400
Virginia Tech contest	3, 54	9, 51	5, 10, 13, 3 others in top 40
Rochester olympiad	1, 3, 5	1,4	1, 1, 5

4 Recruiting

The success of a math team program relies heavily on the quality of students it attracts. An advisor can only do so much as to inspire, encourage, and train his mathletes. Prestigious colleges like Princeton, Harvard, and MIT, have the luxury of recruiting the brightest undergraduates, many of whom were former participants in the International Mathematical Olympiad (IMO). These students, due to their mathematical background, are in a position right from the start to be extremely competitive: in the last three years, at least four of the top 5 individuals on the Putnam exam were previous IMO medalists [1].

At University of Rochester, where we have very good students, but not of this caliber, the challenge was twofold: to keep them interested in the program for a long period of time and to make them believe that they are capable of great performances. In this way, a carefully chosen schedule of advanced classes (e.g., real analysis, abstract algebra, number theory), coupled with an intensive problem solving training, could lead in one or two years to competitive results in these mathematical competitions.

I was fortunate to discover from the very beginning two very good students, who had outstanding work ethics and were committed to an intensive training in problem solving. They had prior experience in mathematical competitions and a strong mathematical background: one of them participated in the Budapest Semester in Mathematics and the Math in Moscow programs, while the other, a Chinese transfer student, had already two years of college courses. They were both members of our 2006 and 2007 Putnam teams. Currently, the former is a doctoral student in mathematics at Dartmouth College, while the latter is pursuing a doctoral degree in economics at Princeton University.

In 2007, I was able to recruit the top high school mathlete in our area, who turned out to be a great addition to our program. In his first year, he excelled in honors classes, usually reserved for juniors and senior math majors, while as a sophomore, he was the captain of our 10th ranked Putnam team. There is sufficient evidence to believe that he will turn into a strong mathematician. My job as an advisor for this program is much easier having enthusiastic students like these ones.

Currently, in an effort to recruit quality mathletes, I have started organizing in fall 2008 the Rochester Area Math Circle. This is an extracurricular activity aimed at mathematically gifted middle school and high school students, which offers an advanced mathematical training that goes beyond the material covered in regular classes. The meetings of the circle are run either as a problem solving seminar or as an expository lecture that presents a specific advanced mathematical topic in an intuitive, interactive, more formal way. My hope is that alumni of this program will choose to come for college at University of Rochester and be the future of our math team.

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